A MODIFIED EXPLICIT P-STABLE TECHNIQUE FOR SOLUTION OF NON-LINEAR EQUATIONS

*Sunila Kashif, Muhammad Naeem Gull, S.Riaz, Saadia Farid
Department of Mathematics,University of Engineering and Technology, Lahore, Pakistan
*Corresponding Author's Email: sunila.kashif@gmail.com

ABSTRACT:- *In this paper, a modified explicit P-Stable technique for simulation of non-linear equations is presented. This technique is bit like Runge-Kutta method. Linear multistep methods examine only at the points t_m = a+mh, m = 0, 1, …, where as Runge-Kutta methods work at off-step points. The direct hybrid methods by Cash [1] and Chawla[2] combine these two features. Methods which are not P-stable can be unreliable for the simulation of fixed step oscillatory problems. A modified explicit P-Stable technique is often essential for the efficient simulation of such problems.*

Key words: Non-linear, P-Stable, Explicit

1. INTRODUCTION

We are concerned with the solution of the system of differential equation which are of order two

$$\frac{d^2\boldsymbol{x}}{dt^2} = f(t, \boldsymbol{x}),\ t \in [a,b]$$

depending on initial conditions

$$\boldsymbol{x}(a) = \boldsymbol{\mu_1},\ \frac{d\boldsymbol{x}}{dt} = \boldsymbol{\mu_2} \qquad (1)$$

where $\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_2}$ are given and $\boldsymbol{x}$, $\mathbf{f}$, $\boldsymbol{\mu_1}, \boldsymbol{\mu_2} \in R^n$.

We have unique solution $\boldsymbol{x}(t)$ of the initial value problem (1).

The direct hybrid method has the form

$$\boldsymbol{x}_{n+1} - 2\boldsymbol{x}_n + \boldsymbol{x}_{n-1} = h^2\{\beta_0 [\boldsymbol{x}''_{n+1} + \boldsymbol{x}''_{n-1}] + \gamma \boldsymbol{x}''_n + \beta_1 [\boldsymbol{x}''_{n+\alpha_1} + \boldsymbol{x}''_{n-\alpha_1}] + \beta_2 [\boldsymbol{x}''_{n+\alpha_2} + \boldsymbol{x}''_{n-\alpha_2}] \qquad (2)$$

$$\boldsymbol{x}_{n\pm\alpha_1} = A_\pm \boldsymbol{x}_{n+1} + B_\pm \boldsymbol{x}_n + C_\pm \boldsymbol{x}_{n-1} + h^2 \{ s_\pm \boldsymbol{x}''_{n+1} + q_\pm \boldsymbol{x}''_n + u_\pm \boldsymbol{x}''_{n-1} \} \qquad (3)$$

$$\boldsymbol{x}_{n\pm\alpha_2} = R_\pm \boldsymbol{x}_{n+1} + L_\pm \boldsymbol{x}_n + T_\pm \boldsymbol{x}_{n-1} + h^2 \{ Y_\pm \boldsymbol{x}''_{n+1} + V_\pm \boldsymbol{x}''_n + W_\pm \boldsymbol{x}''_{n-1} + X_\pm \boldsymbol{x}''_{n-\alpha_1} + Z_\pm \boldsymbol{x}''_{n+\alpha_1}] \} \qquad (4)$$

And $\boldsymbol{x}''_n = \boldsymbol{f}(t_n, \boldsymbol{x}_n)$, $\boldsymbol{x}''_{n\pm1} = \boldsymbol{f}(t_n \pm h, \boldsymbol{x}_{n\pm1})$, $\boldsymbol{x}_{n\pm\alpha_1} = \boldsymbol{f}(t_n+\alpha h, \boldsymbol{x}_{n\pm\alpha_1})$, $\boldsymbol{x}_{n\pm\alpha_2} = \boldsymbol{f}(t_n+\alpha h, \boldsymbol{x}_{n\pm\alpha_2})$ (5)

2. DERIVATION OF PARAMETERS OF SIXTH ORDER DIRECT HYBRID METHOD

Principal local truncation error is given by:

$$\begin{aligned} l[\boldsymbol{x};h]_p = h^6 \{ & [C_1] (\frac{\partial^4 f}{\partial t^4} + 4 \frac{\partial^4 f}{\partial t^3 \partial x}\{\boldsymbol{x}'\} \\ & + 6 \frac{\partial^4 f}{\partial t^2 \partial x^2}\{\boldsymbol{x}'\}^2 + 6 \frac{\partial^3 f}{\partial t^2 \partial x}\{\boldsymbol{x}''\} \\ & +4 \frac{\partial^4 f}{\partial t\, \partial x^3}\{\boldsymbol{x}'\} + 12 \frac{\partial^3 f}{\partial t\, \partial x^2}\{\boldsymbol{x}'\}\{\boldsymbol{x}''\} \\ & + \frac{\partial^4 f}{\partial x^4}\{\boldsymbol{x}'\}^4 + 6 \frac{\partial^3 f}{\partial x^3}\{\boldsymbol{x}'\}^2\{\boldsymbol{x}''\} + 3 \frac{\partial^2 f}{\partial x^2}\{\boldsymbol{x}''\}^2) \\ & + [C_2] (\frac{\partial^2 f}{\partial t\, \partial x}\{\boldsymbol{x}^{(3)}\} + \frac{\partial^2 f}{\partial x^2}\{\boldsymbol{x}'\}\{\boldsymbol{x}^{(3)}\}) \\ & + [C_3] (\frac{\partial f}{\partial x}\{\boldsymbol{x}^{(4)}\}) \\ & + O(h^8) \end{aligned} \qquad (6)$$

where

$$C_1 = -\frac{1}{240} + \frac{1}{12}\beta_1 \alpha_1^2 (1-\alpha_1^2)$$

$$C_2 = -\frac{1}{60} - 2\beta_1 \alpha_1 (s_+ - u_+)$$

$$C_3 = -\frac{1}{240} + \beta_1 \{ \frac{1}{12}(\alpha_1^2 - A_+ - A_-) - (s_+ + s_-) \} \qquad (7)$$

are coefficients of principal local truncation error.

In order to derive parameters of sixth order direct hybrid method, we reduce the coefficients appearing in the principal local truncation error.

We have to Minimize F ($X_1, X_2, \dots, X_n$)

subjects to

$$l_j \le X_j \le u_j, \quad j = 1, 2, \dots, n.$$

Where, l_j and u_j are real and derivatives of F (X) are unavailable.

We define the variables

$$X_1 = \beta_1 (\alpha_1^2 - A_+ - A_-)$$

$$X_2 = \beta_1 (s_+ + s_-) - \frac{X_1}{4}$$

$$X_3 = \frac{1}{12}\beta_1 \alpha_1^2 (1-\alpha_1^2)$$

$$X_4 = 2\beta_1 \alpha_1 (s_+ - u_+)$$

(8)

Conditions are:

$$X_1 \le 0$$

$$X_2 \le 0$$

$$(\frac{2}{3} + 4X_1)^2 < -64X_2 \qquad (9)$$

Coefficients C_1, C_2 and C_3 defined in (5b) may be written as

$$C_1 = -\frac{1}{240} + X_3$$

$$C_2 = -\frac{1}{60} - X_4$$

$$C_3 = -\frac{1}{240} - X_2 - \frac{1}{60}X_1 \qquad (10)$$

We let objective function for the minimization

$$OBJ = C_1^2 + C_2^2 + C_3^2$$

we adopt the following procedure:

I. FIX: X_1; X_1 = 0.0, –0.1, 0.2, … , – 1.0, say,

II. SET: $X_2 < -(2/3 + 4X_1)^2/64$,

III. COMPUTE: X_2, X_3 and X_4 to minimize $C_1^2 + C_2^2 + C_3^2$

Table 1: minimization of OBJ

X_1	X_2	X_3	X_4	C_1	C_2	C_3	OBJ
0.000E + 00	−6.950E-03	4.167E-03	1.677E-02	1.171E-09	1.171E-09	2.783E-03	7.747E-06
1.000E-01	−6.945E-03	3.994E-03	7.118E-02	−3.554E-09	3.554E-09	1.361E-02	4.693E-04
2.000E-01	−2.778E-04	4.167E-03	1.677E-02	−3.562E-09	3.562E-09	2.944E-02	8.670E-04
−3.000E-01	−4.445E-03	4.167E-03	1.677E-02	−9.072E-10	9.072E-10	5.028E-02	2.528E-03
−4.000E-01	−1.362E-02	4.167E-03	1.677E-02	3.594E-09	−3.594E-09	7.611E-02	5.794E-03
−5.000E-01	−2.778E-02	4.167E-03	1.677E-02	−3.606E-09	3.606E-09	1.069E-01	1.144E-02
−6.000E-01	−4.694E-02	4.167E-03	1.677E-02	−3.656E-09	3.656E-09	1.428E-01	2.039E-02
−7.000E-01	−7.111E-02	4.167E-03	1.677E-02	−3.676E-09	3.676E-09	1.8366E-01	3.371E-02
−8.000E-01	−1.003E-01	4.167E-03	1.677E-02	−3.705E-09	3.705E-09	2.294E-01	5.264E-02
−9.000E-01	−1.344E-01	4.167E-03	1.677E-02	−3.618E-09	3.618E-09	2.803E-01	7.856E-02

From Table 1 we note that the objective function OBJ takes its smallest value when $X_1 = 0$.
Now we set,

$C_1 = 0,\ C_2 = 0,\ C_3 = 0$ (11)

With this choice of coefficients we have, from (5b), the following set of parameters for the method:

$$\beta_1 = \frac{1}{20\,\alpha_1^2\,(1-\alpha_1^2)}$$

$$s_+ - u_+ = -\frac{1}{120\,\beta_1\alpha_1}$$

$$s_+ + s_- = \frac{(C_3 + 1/240 - X_1/12)}{\beta_1}$$

$$A_+ + A_- = \alpha_1^2 - \frac{(12\,C_3 + 1/20 + 12\,\beta_1(s_+ + s_-))}{\beta_1} \quad (12)$$

3. NUMERICAL RESULTS

We present some numerical results for the methods mentioned below. We compare our method (Method 3) with following methods:

1) A P-stable, three-evaluation, fourth order method, with reduced truncation error, $A_+ = 0$, $s_- = 0$ and $\alpha_1 = \frac{1}{2}$.
2) A P-stable, three-evaluation, fourth order method, with reduced truncation error, $A_+ = 0$, $s_- = 0$, $\alpha_1 = \frac{1}{2}$ and $u_- = \frac{1}{120\ \beta_1\ \alpha_1}$.
3) A P-stable, three-evaluation, sixth order method.
4) The three-evaluation, fourth order, P-stable method proposed by Cash [4].
5) The perfect square, P-stable, two-evaluation, fourth order method proposed by Thomas [3].
6) A P-stable, two-evaluation, fourth order method, with reduced truncation error.

PROBLEM

$\mathbf{z}''(t) + \mathbf{z}(t) = 0.001\, e^{it}$, $\mathbf{z}(0) = 1$, $\mathbf{z}'(0) = 0.9995i$, $\mathbf{z}(t) \in \Phi$

This linear problem has the analytical solution

$\mathbf{z}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$, $\mathbf{u}(t), \mathbf{v}(t) \in R$

$\mathbf{u}(t) = \cos(t) + 0.0005t\sin(t)$

$\mathbf{v}(t) = \sin(t) - 0.0005t\cos(t)$

We compare the performance of above mentioned methods with step sizes h = $\pi/4, \pi/5, \pi/6, \pi/8, \pi/9, \pi/12, \pi/16$ and $\pi/32$.

Table 2: Solutions of Method 1

h	Numerical value	Error
$\pi/4$	1.0039932	2.0211819E-03
$\pi/5$	1.0028050	8.3298147E-04
$\pi/6$	1.0023758	4.0380182E-04
$\pi/8$	1.0021005	1.2855861E-04
$\pi/9$	1.0020524	8.0403617E-05
$\pi/12$	1.0019975	2.5518384E-05
$\pi/16$	1.0019801	8.0884298E-06
$\pi/32$	1.0019725	5.0639475E-07

Table 3: Solutions of Method 2

h	Numerical value	Error
$\pi/4$	1.0039929	2.0209846E-03
$\pi/5$	1.0028048	8.3287362E-04
$\pi/6$	1.0023757	4.0375003E-04
$\pi/8$	1.0021005	1.2854226E-04
$\pi/9$	1.0020524	8.0393410E-05
$\pi/12$	1.0019975	2.5515156E-05
$\pi/16$	1.0019801	8.0874083E-06
$\pi/32$	1.0019725	5.0633088E-07

Table 4: Solutions of Method 3

h	Numerical value	Error
$\pi/4$	1.0039942	7.7816499E-05
$\pi/5$	1.0019518	2.0224547E-05
$\pi/6$	1.0019652	6,7423578E-06
$\pi/8$	1.0019708	1.1947001E-06
$\pi/9$	1.0019714	5.8861967E-07
$\pi/12$	1.0019719	1.0456173E-07
$\pi/16$	1.0019720	1.8589973E-08
$\pi/32$	1.0019720	2.9017921E-10

Table 5: Solutions of Method 4

h	Numerical value	Error
$\pi/4$	1.0041184	2.1463791E-03
$\pi/5$	1.0028560	8.8404347E-04
$\pi/6$	1.0024004	4.2842206E-04
$\pi/8$	1.0021083	1.3635357E-04
$\pi/9$	1.0020572	8.5271031E-05
$\pi/12$	1.0019990	2.7058893E-05
$\pi/16$	1.0019805	8.0884298E-06
$\pi/32$	1.0019725	5.3668365E-07

Table 6: Solutions of Method 5

h	Numerical value	Error
$\pi/4$	1.0475928	4.5620792E-02
$\pi/5$	1.0353402	3.3368233E-02
$\pi/6$	1.0189177	1.6945725E-02
$\pi/8$	1.0075474	5.5754652E-03
$\pi/9$	1.0055154	3.5434628E-03
$\pi/12$	1.0031348	1.1628158E-03
$\pi/16$	1.0023487	3.7668894E-04
$\pi/32$	1.0019961	2.4116392E-05

Table 7: Solutions of Method 6

h	Numerical value	Error
$\pi/4$	1.0039932	2.0211819E-03
$\pi/5$	1.0028050	8.3298147E-04
$\pi/6$	1.0023758	4.0380182E-04
$\pi/8$	1.0021005	1.2855861E-04
$\pi/9$	1.0020524	8.0403617E-05
$\pi/12$	1.0019975	2.5518384E-05
$\pi/16$	1.0019801	8.0884298E-06
$\pi/32$	1.0019725	5.0639475E-07

4. CONCLUSION

. On comparing these results, we note that the P-stable fourth order methods with reduced truncation error, that is, methods (1), (2), (3), (5) and (6) are slightly more accurate than Cash's method (method (4)). There is a marginal difference between the accuracy of Methods (1) and (2), while methods (1) and (6) have same accuracy. The sixth order method, that is, method (3), is more accurate than any of the five methods, while method (5) is the least accurate method.

The derivation of higher order direct hybrid methods with reduced truncation error and the application of our new methods in variable step codes is included in future targets.

REFERENCES:

[1] **Cash, J. R.** (1981) High order P-stable Formulae for the Numerical Integration of Periodic Initial Value Problems. Numer. Math., **37**, 355--370.

[**2**] **Chawla, M. M.** (1981) Two-step fourth order P-stable methods for Second Order Differential Equations, BIT, **21**, 190--193.

[3] **Cash, J. R.** (1981) High order P-stable Formulae for the Numerical Integration of Periodic Initial Value Problems. Numer. Math., **37**, 355--370.

[4] Chawla, M. M. (1981) Two-step fourth order P-stable methods for Second Order Differential Equations, BIT, **21**, 190--193.

[5] Thomas, R. M. (1987) Efficient Fourth Order P-stable Formulae, BIT, 27, 599-614.

[6] Henrici, P. (1962) Discrete Variable Methods for Ordinary Differential Equations, John Wiley, New York.